

Sublinear elliptic problems with a Hardy potential.

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Abstract

In this paper we study the positive solutions of sub linear elliptic equations with a Hardy potential which is singular at the boundary. By means of ODE techniques a fairly complete picture of the class of radial solutions is given. Local solutions with a prescribed growth at the boundary are constructed by means of contraction operators. Some of those radial solutions are then used to construct ordered upper and lower solutions in general domains. By standard iteration arguments the existence of positive solutions is proved. An important tool is the Hardy constant.

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1 Introduction

In this paper we study positive solutions of problems of the form

$$\Delta u + \frac{\mu}{\delta(x)^2} u = u^p \text{ in } \Omega, \quad (1.1)$$

where $\mu \in \mathbb{R} \setminus \{0\}$, $\delta(x)$ is the distance of a point $x \in \Omega$ to the boundary, $0 < p < 1$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded, smooth domain. The expression

$$\frac{\mu}{\delta(x)^2} =: V_\mu(x)$$

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is called the *Hardy potential*. In this type of problems there are two competing mechanisms, namely the nonlinear problem

$$\Delta u = u^p \text{ in } \Omega, \quad (1.2)$$

and the linear problem

$$\Delta h + V_\mu(x)h = 0 \text{ in } \Omega, \quad (1.3)$$

The problem (1.2) is well-understood cf. [5], [4]. For any continuous function $\phi \geq 0$ it has a unique solution with $u = \phi$ on the boundary. Moreover if ϕ is small or if the domain is large the solutions have a *dead core*, i.e. an open set $\omega \in \Omega$ where the solution vanishes identically. For the linear problem (1.3) boundary values cannot be prescribed arbitrarily because of the singularity of the Hardy potential.

The case $p > 1$ has been studied in [2]. There among others, a partial classification of the solutions has been given. It is based on the simple observation that the solutions of (1.1) are lower solutions for the linear problem, and on some results of their local behavior near the boundary [1]. Another related study where the nonlinearity is the exponential function e^u has been carried out in [3].

It turns out that (1.1) has many solutions. We start with the investigation of radial solutions and provide a fairly complete picture of their structure. There are solutions whose boundary behavior is determined by the nonlinearity (1.2) and others by the Hardy potential (1.3). In this latter case solutions have the same behavior as the harmonics of the 1- dimensional problem $h'' + V_\mu(x)h = 0$ in $(-L, L)$. Indeed the corresponding indicial equation is

$$\beta(\beta - 1) + \mu = 0. \quad (1.4)$$

Hence positive harmonics near $x = -L$ and $x = L$ exist if and only if $\mu \leq 1/4$. Set

$$\beta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}. \quad (1.5)$$

In this case $\delta(x) = L - |x|$ and the harmonics are of the form ($x \in (-L, 0)$ or $x \in (0, L)$ for any given constants $c_1, c_2 \in \mathbb{R}$)

$$h(x) = c_1 \delta^{\beta_+} + c_2 \delta^{\beta_-},$$

provided $\beta_- \neq \beta_+$. Otherwise

$$h(x) = c_1 \delta^{1/2} + c_2 \delta^{1/2} \log \frac{1}{\delta}.$$

Observe that the derivative h' is not in L^2 if c_2 is different from zero.

We will show that there are only three possible boundary behaviors for the positive radial solutions, namely

1. $\lim_{\delta \rightarrow 0} \frac{u(\delta)}{\delta^{2/(1-p)}} = c',$ (nonlinear regime)
2. $\lim_{\delta \rightarrow 0} \frac{u(\delta)}{\delta^{\beta_-}} = c_1,$ (linear singular regime)
3. $\lim_{\delta \rightarrow 0} \frac{u(\delta)}{\delta^{\beta_+}} = c_2$ (linear regular regime) .

If the order of the linear regular regime is higher then the order of the nonlinear regime, only the second case occurs. We shall also prove the existence of local solutions with the boundary behavior described above. These solutions are then used to construct upper and lower solutions in general domains.

An important tool for proving the existence of global solutions is the *Hardy constant*. It is defined as

$$C_H(\Omega) = \inf_{\phi \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^2 dx}{\int_{\Omega} \delta^{-2}(x) \phi^2 dx}. \quad (1.6)$$

It is well-known that $0 < C_H \leq 1/4$ and $C_H(\Omega) = 1/4$ for convex domains and for annuli if $N > 2$, see Marcus, Mizel and Pinchover [6]. If $N = 2$ they proved that $C_H \rightarrow 0$ if the outer radius tends to infinity. They also showed that for thin parallel sets $C_H = 1/4$ and that the Hardy constant is attained if and only if $C_H < 1/4$.

In our investigations the following *comparison principle* will play an important role:

Let $\mu < C_H(\Omega)$ and $\omega \subseteq \Omega$. If $\Delta u + V_{\mu}u \geq 0$ in ω and $u \in W_0^{1,2}(\omega)$ then $u \leq 0$ in ω .

In fact u^+ is an admissible function for (1.6). Testing the inequality $\Delta u + V_{\mu}u \geq 0$ with u^+ we obtain $-\int_{\Omega} |\nabla u^+|^2 dx + \mu \int_{\Omega} \frac{(u^+)^2}{\delta^2} dx \geq 0$. Hence $\mu \geq C_H(\Omega)$ which contradicts our assumption.

Our paper is organized as follows. We first study the radial solutions in balls and annuli. By means of ODE techniques we discuss the existence of local solutions with and without dead core and we show how to continue them globally. We then determine their asymptotic behavior near the boundary. At the end we prove the existence of positive solutions in arbitrary domains.

2 Radial solutions, local behavior

2.1 Local solutions

In this section we study the radial solutions $u(r)$, $r = |x|$, of (1.1) in balls B_R of radius R , centered at the origin, and in annuli $\mathcal{A}(r_0, R) = \{x : r_0 < |x| < R\}$, $r_0 > 0$. They satisfy the ordinary differential equation

$$u'' + \frac{(N-1)}{r}u' + \frac{\mu}{\delta(r)^2}u = u^p \text{ where } r \in (0, R) \text{ or } r \in (r_0, R). \quad (2.1)$$

Here $u'(r) := \frac{d}{dr}u(r)$. It is well-known that problem (2.1) with the initial conditions

$$u(0) = u_0 > 0, u'(0) = 0 \quad (2.2)$$

or

$$u(R_0) = u_0 > 0, u'(R_0) = u_1 \in \mathbb{R} \text{ for } R > R_0 > r_0 > 0 \quad (2.3)$$

has a unique local solution which is positive in a neighborhood of $r = 0$ or of R_0 , respectively. Since the nonlinearity is not Lipschitz continuous at $u = 0$, the trivial solution is not the only solution with $u(R_0) = 0$ and $u'(R_0) = 0$. In fact we shall prove that there exists a local solution such that for a given $R_0 \geq 0$ we have $u(R_0) = 0$, $u'(R_0) = 0$ and $u > 0$ for $r > R_0$ and/or for $r < R_0$.

2.1.1 Solutions with a dead core

In our investigations there is a critical value of μ which will play an essential role. Define

$$\mu^* := \frac{2(p+1)}{(1-p)^2}. \quad (2.4)$$

Lemma 2.1 (i) Let R_0 be a given point in (r_0, R) in the case of an annulus, or in $(0, R)$ in the case of a ball. Then in a small neighborhood of R_0 there exists a positive solution of (2.1) which is of the form $u(r) = |r - R_0|^{\frac{2}{1-p}}(c_p + w(r - R_0))$ and has the property that $u(R_0) = u'(R_0) = 0$. Moreover $w(0) = 0$ and $c_p = (\mu^*)^{\frac{1}{1-p}}$.

(ii) If $R_0 = r_0 > 0$ or $R_0 = R$, then the same statement holds true provided $\mu > -\mu^*$. In this case c_p has to be replaced by $c' = (\mu^* + \mu)^{\frac{1}{p-1}}$.

(iii) In the ball, near the origin, there exists a local solution of the form $u(r) = r^{\frac{2}{1-p}}(c'' + w(r))$ with $c'' = (\mu^* + \frac{2(N-1)}{1-p})^{\frac{1}{p-1}}$ and $w(0) = 0$.

Proof. Let us introduce in (2.1) the new variable $d = r - R_0$. Then (2.1) assumes the form

$$u'' + \frac{N-1}{R_0+d}u' + \frac{\mu}{\delta^2}u = u^p \text{ in } (r_0 - R_0, R - R_0).$$

For simplicity we shall write $u(d)$ for $u(R_0+d)$. Assuming that $u(0) = u'(0) = 0$ we obtain after integration

$$u(d) = \int_0^d \sigma(s) \left(u^p - \frac{\mu}{\delta^2} u \right) \left(\int_s^d \frac{dt}{\sigma(t)} \right) ds,$$

where

$$\sigma(d) = (R_0 + d)^{N-1}.$$

Then

$$u(d) = \int_0^d K_N(s, d; R_0) \left[u^p - \frac{\mu}{\delta^2} u \right] ds \quad (2.5)$$

where

$$\begin{aligned} K_1 &= d - s && \text{if } N = 1, \\ K_2 &= (R_0 + s) \ln \left(\frac{R_0 + d}{R_0 + s} \right) && \text{if } N = 2, \\ K_N &= \frac{R_0 + s}{N-2} \left[1 - \left(\frac{R_0 + s}{R_0 + d} \right)^{N-2} \right] && \text{if } N > 2. \end{aligned}$$

The distance expressed in the variable d becomes

$$\delta(d) = \begin{cases} R - R_0 - d & \text{if } R_0 + d > (R + r_0)/2, \\ R_0 - r_0 + d & \text{if } R_0 + d < (R + r_0)/2. \end{cases}$$

From the Taylor expansion we obtain

$$K_N(s, d; R_0) = d - s + O((d - s)^2). \quad (2.6)$$

Observe that (2.5) is also defined for negative d . Set

$$u(d) := |d|^{\frac{2}{1-p}} (c_p + w(d)). \quad (2.7)$$

By (2.5) we have $w(d) = (Tw)(d)$ where

$$(Tw)(d) := \frac{1}{|d|^{\frac{2}{1-p}}} \int_0^d K_N(s, d) \left[|s|^{\frac{2p}{1-p}} (c_p + w)^p - \frac{\mu}{\delta^2} |s|^{\frac{2}{1-p}} (c_p + w) \right] ds - c_p. \quad (2.8)$$

Tw is well defined. Indeed by (2.6) we obtain for the lowest order term in the integral

$$\frac{1}{|d|^{\frac{2}{1-p}}} \int_0^d (d-s)|s|^{\frac{2p}{1-p}} ds = c_p^{1-p}. \quad (2.9)$$

Next we want to show that Tw has a fixed point. Fix α such that

$$p < \alpha < 1, \quad (2.10)$$

and define

$$M := c_p \left(1 - \left(\frac{p}{\alpha}\right)^{\frac{1}{1-p}}\right) < c_p < 1 \text{ and } X := \{w \in C^0([-d_0, d_0]) : |w|_\infty \leq M\},$$

where $d_0 \in (0, d_0^*]$ for some d_0^* such that $\delta(\pm d_0^*) > 0$. From the definition of M it follows that

$$\alpha = \frac{pc_p^{1-p}}{(c_p - M)^{1-p}} < 1. \quad (2.11)$$

The following two properties hold:

- (i) T is a contraction in X . A direct computation shows that for $d_0 \in (0, d_0^*]$

$$0 \leq \int_0^d K_N(s, d) ds \leq C_0 d^2 \text{ where } C_0 \text{ is independent of } d_0.$$

This together with (2.6), (2.9), (2.8) and (2.11) implies that, for sufficiently small d_0 , and for a constant C_1 independent of d_0

$$\begin{aligned} |Tw_1 - Tw_2| &\leq \frac{pc_p^{1-p}}{(c_p - M)^{1-p}} |w_1 - w_2|_\infty + C_1 |d| |w_1 - w_2|_\infty \\ &= (\alpha + C_1 |d|) |w_1 - w_2|_\infty \leq \frac{\alpha + 1}{2} |w_1 - w_2|_\infty. \end{aligned} \quad (2.12)$$

- (ii) $T : X \rightarrow X$. For any $w \in X$ we have from the previous estimate

$$\begin{aligned} |Tw(\xi)| &\leq |Tw(\xi) - T0| + |T0| \\ &\leq \frac{\alpha + 1}{2} |w|_\infty + C_1 c_p |d| \\ &\leq \frac{\alpha + 1}{2} M + C_1 c_p |d| \leq M, \end{aligned} \quad (2.13)$$

for $|d| \leq d_0$ sufficiently small.

Notice that by the special choice of c_p , the fixed point satisfies $w(0) = 0$ and consequently u is positive in a neighborhood of R_0 .

If $R_0 = R$ or r_0 , then $\delta(s) = |s|$, thus the linear term in (2.8) is of the form $\mu|s|^{\frac{2p}{1-p}}(c' + w)$. In order to have $w(0) = 0$ we have to choose c' suitably. With this change the remainder of the proof is the same as before. Similarly in the ball we have to adjust the constant if $R_0 = 0$. The details will be omitted. This concludes the proof of the lemma. \square

In the previous lemma we have constructed a solution which vanishes together with its derivative at one point R_0 . This solution gives rise to other solutions.

Corollary 2.1 *For any $r_0 < R'_0 \leq R_0 < R$, (2.1) has a solution which is positive in $(R'_0 - \epsilon, R'_0) \cup (R_0, R_0 + \epsilon)$ for $\epsilon > 0$ sufficiently small and which vanishes in $[R'_0, R_0]$. We say that it has a dead core in $[R'_0, R_0]$. Moreover there exist solutions vanishing in (r_0, R_0) or in (R_0, R) and positive in $(R_0, R_0 + \epsilon)$ or in $(R_0 - \epsilon, R_0)$.*

Corollary 2.2 *Assume $\mu \in (-\mu^*, \frac{1}{4})$, $\mu \neq 0$. If u is a local solution satisfying $\lim_{\delta \rightarrow 0} \frac{u(\delta)}{\delta^{\frac{2}{1-p}}} = 0$ then $u \equiv 0$ in some neighborhood of the boundary.*

Proof. By contradiction suppose that there exists such a solution u which is positive in $(0, \delta_0]$ ($\delta_0 > 0$).

We assume $\delta_0 > 0$ so small that $C_H(\mathcal{A}(R - 2\delta_0, R)) = \frac{1}{4}$, and $C_H(\mathcal{A}(r_0, r_0 + 2\delta_0)) = \frac{1}{4}$. Then the maximum principle holds also if we are working in a larger annulus, if we deal with functions which belong to $W_0^{1,2}(\mathcal{A}(R - 2\delta_0, R))$ or $W_0^{1,2}(\mathcal{A}(r_0, r_0 + 2\delta_0))$.

First assume $\mu \in (0, \frac{1}{4})$. Let \tilde{u} be the solution constructed in Lemma 2.1, (ii). Since $\lim_{\delta \rightarrow 0} \frac{\tilde{u}(\delta)}{\delta^{\frac{2}{1-p}}} = c' = (\mu^* + \mu)^{\frac{1}{p-1}} > 0$, we have

$$u(\delta) < \tilde{u}(\delta), \quad \forall \delta \in (0, \delta_0], \quad (2.14)$$

for a possibly smaller $\delta_0 > 0$. For any $\epsilon \in (0, \delta_0)$, consider the function $\tilde{u}_\epsilon(\delta)$ such that $\tilde{u}_\epsilon(\delta) = 0$ in $[0, \epsilon]$, and $\tilde{u}_\epsilon(\delta) = \tilde{u}(\delta - \epsilon)$ in $(\epsilon, \delta_0 - \epsilon]$. By (2.14) there exists for sufficiently small ϵ , a number $\delta_1 \in (0, \delta_0]$ such that $\tilde{u}_\epsilon(\delta) < u(\delta)$ in $(0, \delta_1)$ and $\tilde{u}_\epsilon(\delta_1) = u(\delta_1)$. Since we have assumed that $\mu > 0$ and since \tilde{u}_ϵ belongs to $C^1([0, \delta_0])$, it can easily be seen that \tilde{u}_ϵ is an upper solution of (1.1), both near the inner or outer boundary. Then $u - \tilde{u}_\epsilon = 0$ at $\delta = 0$ and $\delta = \delta_1$, $u - \tilde{u}_\epsilon > 0$ and satisfies $\Delta(u - \tilde{u}_\epsilon) + \frac{\mu}{\delta(x)^2}(u - \tilde{u}_\epsilon) \geq u^p - \tilde{u}_\epsilon^p \geq 0$ for $0 < \delta < \delta_1$.

This contradicts the maximum principle. Consequently u vanishes in a neighborhood of zero.

If $\mu \in (-\mu^*, 0)$, let $0 < \epsilon < \delta_0$ and let C be a positive number. Consider the function $z_\epsilon(\delta) = 0$ in $[0, \epsilon]$, and $z_\epsilon(\delta) = C(\delta - \epsilon)^{\frac{2}{1-p}}$, for $\delta \in (\epsilon, \delta_0]$. z_ϵ is $C^1([0, \delta_0])$ and for $\delta \in (\epsilon, \delta_0]$ it satisfies

$$\begin{aligned} \Delta z_\epsilon + \frac{\mu}{\delta^2} z_\epsilon - z_\epsilon^p &< \Delta z_\epsilon - z_\epsilon^p \\ &= C \delta^{\frac{2p}{1-p}} \left[\frac{2(p+1)}{(1-p)^2} + \frac{2(N-1)}{r(\delta)(1-p)} (\delta - \epsilon) - C^{p-1} \right], \end{aligned} \quad (2.15)$$

where $r(\delta) = R - \delta$ at the outer boundary, $r(\delta) = r_0 + \delta$ at the inner boundary. There exists a small positive constant C_0 depending only on δ_0 such that the expression in the brackets of (2.15) is negative for all $\delta \in (\epsilon, \delta_0]$, hence z_ϵ is an upper solution for $C = C_0$. Because of our assumption we have $u \leq \frac{C_0}{2} \delta^{\frac{2}{1-p}}$ in $(0, \delta_0]$ (for a possibly smaller δ_0).

Next we determine ϵ such that $z_\epsilon(\delta_0) \geq \frac{C_0}{2} \delta_0^{\frac{2}{1-p}}$. Then there exists $\delta_1 \leq \delta_0$ such that $z_\epsilon(\delta) < u(\delta)$ in $(0, \delta_1)$ and $z_\epsilon(\delta_1) = u(\delta_1)$. This is impossible by the comparison principle, as in the case of positive μ . Consequently u vanishes in a neighborhood of zero. \square

2.1.2 Continuation of local solutions

Consider a local solution u of the initial value problem (2.1), (2.2) or (2.1), (2.3) respectively. This solution can be continued up to the boundary unless it vanishes or blows up at an interior point. Blowup can be excluded because the nonlinearity is sub linear.

Consider first a ball B_R . Assume that u satisfies (2.1), (2.2) and $\mu < \frac{1}{4}$. Then by the comparison principle stated in the Introduction u cannot vanish at an inner point. Hence it can be continued as a global solution up to the boundary. By the same argument we can show that a solution with a dead core can be continued as a positive solution up to the boundary. The positive solution to the left ($r < R_0$) can be continued up to the origin but it is singular at the origin.

Consider now the solution of (2.1), (2.3) in an annulus and let $\mu < C_H(\mathcal{A}(r_0, R))$. The solution can be continued at both sides until it vanishes or it reaches the boundary. By the comparison principle it cannot vanish at both sides at an interior point. Hence at least at one side it reaches the boundary. Thus a solution with a dead core can be continued as a positive solution which does not vanish at an inner point.

2.2 Asymptotic behavior at the boundary

In this section we assume that there exists a positive solution up to the inner or outer boundary in an annulus, and we want to determine the asymptotic behavior of u as $r \rightarrow R$ or $r \rightarrow r_0$. The results in a ball coincide with those at the outer boundary of an annulus.

THROUGHOUT THIS SECTION WE SHALL ASSUME THAT $\mu < \frac{1}{4}$ AND $\mu \neq 0$.

The case $\mu = \frac{1}{4}$ can be treated similarly, but requires some further arguments and will therefore be omitted.

For this purpose we choose the distance from the boundary δ instead of r as the new variable and we write $u = \delta^\beta v$ where $\beta = \beta_+$ or β_- defined in (1.5), i.e.

$$\beta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}.$$

From (2.1), for $\delta \in (0, \frac{R+r_0}{2})$ we obtain

$$v'' + \left(2\frac{\beta}{\delta} - \frac{N-1}{R-\delta}\right)v' - \beta\frac{N-1}{(R-\delta)\delta}v = v^p\delta^{\beta(p-1)} \text{ if } \delta = R - r, \quad (2.16)$$

$$v'' + \left(2\frac{\beta}{\delta} + \frac{N-1}{r_0+\delta}\right)v' + \beta\frac{N-1}{(r_0+\delta)\delta}v = v^p\delta^{\beta(p-1)} \text{ if } \delta = r - r_0.$$

These equations can be written in the form

$$(\sigma_- v')' = \sigma_- \left(v^p \delta^{\beta(p-1)} + \beta \frac{N-1}{(R-\delta)\delta} v \right), \text{ where } \sigma_-(\delta) = \delta^{2\beta} (R-\delta)^{N-1}, \quad (2.17)$$

$$(\sigma_+ v')' = \sigma_+ \left(v^p \delta^{\beta(p-1)} - \beta \frac{N-1}{(r_0+\delta)\delta} v \right), \text{ where } \sigma_+(\delta) = \delta^{2\beta} (r_0+\delta)^{N-1} \quad (2.18)$$

Lemma 2.2 *Let v be a solution of (2.17) with $\beta = \beta_- > 0$ or of (2.18) with $\beta = \beta_- < 0$. Then*

$$\lim_{\delta \rightarrow 0} v(\delta) = v(0) < \infty.$$

Proof. From the differential equations (2.17) and (2.18) it follows immediately that for our particular choice of β , v has no local maximum. It is therefore monotone near zero, hence there exists $\lim_{\delta \rightarrow 0} v(\delta) =$

$v(0)$. Next we want to show that $v(0) < \infty$. Suppose on the contrary that $v(0) = \infty$. Integration of (2.17) yields

$$v(\delta) - v(\delta_0) + \sigma_-(\delta_0)v'(\delta_0) \int_{\delta}^{\delta_0} \sigma_-^{-1} ds = \int_{\delta}^{\delta_0} \sigma_- (v^p s^{\beta(p-1)} + \beta \frac{N-1}{(R-s)s} v) ds \int_{\delta}^s \sigma_-^{-1} d\xi.$$

For $s \leq \delta_0$ we have since $\beta < 1/2$

$$\int_{\delta}^s \sigma_-^{-1} d\xi \leq \frac{s^{1-2\beta}}{(1-2\beta)(R-\delta_0)^{N-1}}.$$

Since by assumption $v(\delta)$ is mono tone increasing near the origin

$$v(\delta) \leq v(\delta_0) + c_1 v'(\delta_0) \delta_0 + v^p g(\delta_0) + \frac{\beta(N-1)R^{N-1}}{(R-\delta_0)^N(1-2\beta)} \delta_0 v(\delta),$$

where c_1 and $g(\delta_0)$ are independent of δ . We now choose δ_0 so small that

$$v(\delta) \leq v(\delta_0) + c_1 v'(\delta_0) \delta_0 + v^p g(\delta_0) + \epsilon v(\delta) \text{ for } \epsilon < 1.$$

From here we deduce that $v(0) < \infty$. The same argument applies to the second statement. \square

If $\beta = \beta_-$ is of opposite sign the statement remains true but a different argument is required.

Lemma 2.3 *Let v be a solution of (2.17) with $\beta = \beta_- < 0$ or of (2.18) with $\beta = \beta_- > 0$. Then*

$$\lim_{\delta \rightarrow 0} v(\delta) = v(0) < \infty.$$

Proof. (2.17) and (2.18) imply that

$$(\sigma_{\pm} v')' \leq \sigma_{\pm} v^p \delta^{\beta(p-1)}.$$

Hence

$$v(\delta) \leq v(\delta_0) - \sigma_{\pm}(\delta_0)v'(\delta_0) \int_{\delta}^{\delta_0} \sigma_{\pm}^{-1}(s) ds + \int_{\delta}^{\delta_0} v^p \xi^{\beta(p-1)} \sigma_{\pm} d\xi \int_{\delta}^{\xi} \sigma_{\pm}^{-1}(t) dt,$$

where $\sigma_-(\delta) = \delta^{2\beta}(R-\delta)^{N-1}$ and $\sigma_+(\delta) = \delta^{2\beta}(r_0+\delta)^{N-1}$.

Since $1 - 2\beta > 0$ and $\beta(p - 1) + 1 > 0$ it follows that

$$v(\delta) \leq C_1 + C_2 \int_{\delta}^{\delta_0} v^p d\xi, \text{ where } C_1, C_2 \text{ are independent of } \delta.$$

From this inequality we deduce that v is uniformly bounded.

Next we want to show that $v(\delta)$ has a limit as δ tends to 0.

A. We first consider the equation (2.18) with $\beta > 0$. After integration we obtain

$$\sigma_+(\delta)v'(\delta) - \sigma_+(\epsilon)v'(\epsilon) = \int_{\epsilon}^{\delta} \sigma_+ \left(v^p s^{\beta(p-1)} - \beta \frac{(N-1)}{(r_0+s)s} v \right) ds \quad (2.19)$$

Notice that the right hand integral converges as $\epsilon \rightarrow 0$. We now distinguish between two cases.

1. $\lim_{\epsilon \rightarrow 0} \sigma_+(\epsilon)v'(\epsilon) = 0$. Then

$$\sigma_+(\delta)v'(\delta) = \int_0^{\delta} \sigma_+ \left(v^p s^{\beta(p-1)} - \beta \frac{(N-1)}{(r_0+s)s} v \right) ds.$$

Since v is bounded the following estimate holds true

$$|v'(\delta)| \leq \frac{M}{\delta^{2\beta}} \int_0^{\delta} s^{2\beta} \left(s^{\beta(p-1)} + \frac{1}{s} \right) ds \leq c_1 \delta^{1+\beta(p-1)} + c_2,$$

where c_1, c_2 are independent of δ . Since $\beta < \frac{1}{2}$, $|v'|$ is bounded and hence $\lim_{\delta \rightarrow 0} v(\delta) = v(0)$.

2. $\lim_{\epsilon \rightarrow 0} \sigma_+(\epsilon)v'(\epsilon) = L \neq 0$. Then $v'(\delta) \rightarrow \pm\infty$ as $\delta \rightarrow 0$, depending on the sign of L . Again since $\beta_- < \frac{1}{2}$, $v(\delta)$ has a finite limit as $\delta \rightarrow 0$.

B. Consider (2.17) with $\beta < 0$. As before we integrate (2.17) and find

$$\sigma_-(\delta)v'(\delta) - \sigma_-(\epsilon)v'(\epsilon) = \int_{\epsilon}^{\delta} \sigma_- \left(v^p s^{\beta(p-1)} + \beta \frac{N-1}{(R-s)s} v \right) ds.$$

Dividing this expression by $\sigma_-(\epsilon)$ we obtain the estimate

$$|v'(\epsilon)| \leq c_1(\delta)\epsilon^{-2\beta} + c_2|v|_{\infty}^p \epsilon^{-\beta(1-p)+1} + c_3|v|_{\infty},$$

where c_1, c_2, c_3 are independent of ϵ . Consequently $|v'|$ is bounded and the $\lim_{\delta \rightarrow 0} v(\delta)$ exists. This completes the proof of the lemma. \square

As a consequence we can determine more precisely the behavior of $v(\delta)$ near zero.

Proposition 2.1 *Let v be a solution of (2.17) or (2.18). Then*

(i) If $\beta_- > 0$ then $\lim_{\epsilon \rightarrow 0} \sigma_{\pm}(\epsilon)v'(\epsilon) = L$.

Moreover if $L = 0$ then

$$v'(0) = \frac{N-1}{2R}v(0) \text{ at the outer boundary } r = R \quad (2.20)$$

and

$$v'(0) = -\frac{N-1}{2r_0}v(0) \text{ at the inner boundary } r = r_0. \quad (2.21)$$

(ii) If $\beta_- < 0$ then (2.20) and (2.21) hold.

Proof. By Lemmas 2.2, 2.3, we have that v is continuous up to the boundary, hence it is bounded. Near the outer boundary the function v satisfies

$$\begin{aligned} & \sigma_-(\delta)v'(\delta) - \sigma_-(\epsilon)v'(\epsilon) \\ &= \int_{\epsilon}^{\delta} \sigma_- \left(v^p s^{\beta(p-1)} + \beta \frac{N-1}{(R-s)s} v \right) ds, \end{aligned} \quad (2.22)$$

where $\sigma_-(\delta) = \delta^{2\beta}(R-\delta)^{N-1}$. If $\beta_- > 0$ the limit as ϵ tends to zero exists and is bounded, as we have already remarked. Let $\lim_{\epsilon \rightarrow 0} \sigma_-(\epsilon)v'(\epsilon) = L$. If $L = 0$ then

$$\sigma_-(\delta)v'(\delta) = \int_0^{\delta} \sigma_- \left(v^p s^{\beta(p-1)} + \beta \frac{N-1}{(R-s)s} v \right) ds.$$

Dividing by $\sigma_-(\delta)$ and applying the rule of Bernoulli l'Hospital we obtain (2.20).

If $\beta_- < 0$ and $v(0) > 0$, the integral at the right-hand side of (2.22) becomes infinite as $\epsilon \rightarrow 0$. If we divide by $\sigma_-(\epsilon)$ and apply the rule of Bernoulli l'Hospital the assertion follows. If $v(0) = 0$ the same proof works if the integral at the right-hand side of (2.22) diverges for $\epsilon \rightarrow 0$. If $v(0) = 0$ and we have no information on the behavior of the integral as $\epsilon \rightarrow 0$, from (2.22) we get

$$\begin{aligned} & \frac{\sigma_-(\delta)v'(\delta) - \int_{\epsilon}^{\delta} \sigma_- v^p s^{\beta(p-1)} ds}{\sigma_-(\epsilon)} \leq v'(\epsilon) \\ & \leq \frac{\sigma_-(\delta)v'(\delta) + |\beta| \int_{\epsilon}^{\delta} \sigma_- \frac{N-1}{(R-s)s} v ds}{\sigma_-(\epsilon)}. \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$, we get $v'(0) = 0$. Indeed the upper and lower bound are quotients. The denominator diverges, hence the limit

is zero if the numerator is bounded. If the numerator is unbounded the application of the rule of Bernoulli l'Hospital gives the result. This completes the proof for the outer boundary.

The arguments in the case of the inner boundary are exactly the same. \square

The main result of this section is summarized in

Corollary 2.3 *Suppose that u exists and is positive up to the boundary. Assume $\mu < \frac{1}{4}$, $\mu \neq 0$, and let $\beta_- = \frac{1}{2} - \sqrt{\frac{1}{4} - \mu}$. Then*

$$\frac{u(\delta)}{\delta^{\beta_-}} \rightarrow v(0) \text{ as } \delta \rightarrow 0.$$

Next we want to know more precisely what happens if $\frac{u(\delta)}{\delta^{\beta_-}} \rightarrow 0$ as $\delta \rightarrow 0$.

Lemma 2.4 *Assume $\mu < \frac{1}{4}$, $\mu \neq 0$ and $\frac{u(\delta)}{\delta^{\beta_-}} \rightarrow 0$ as $\delta \rightarrow 0$. Then there exists a nonnegative constant $c \geq 0$ such that*

$$u(\delta) \leq c\delta^{\beta_+}$$

in a neighborhood of the boundary.

Proof. Set $u = \delta^\beta w$ where $\beta = \beta_+$. Then condition $\frac{u(\delta)}{\delta^{\beta_-}} \rightarrow 0$ as $\delta \rightarrow 0$ is equivalent to

$$w(\delta) = \delta^{1-2\beta}v(\delta), \quad \text{where } v(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (2.23)$$

The functions w and v are solutions of (2.16) with $\beta = \beta_+$ or β_- , respectively. Integrating (2.17), or (2.18), respectively in $[\epsilon, \delta]$ for v replaced by w , we obtain

$$\sigma_-(\delta)w'(\delta) - \sigma_-(\epsilon)w'(\epsilon) = \int_\epsilon^\delta \sigma_-(w^p s^{\beta(p-1)+\beta} \frac{N-1}{(R-s)s} w) ds \quad (2.24)$$

and

$$\sigma_+(\delta)w'(\delta) - \sigma_+(\epsilon)w'(\epsilon) = \int_\epsilon^\delta \sigma_+(w^p s^{\beta(p-1)-\beta} \frac{N-1}{(r_0+s)s} w) ds \quad (2.25)$$

Since $\sigma_\pm \sim \delta^{2\beta}$ near zero it follows from our assumption that $\sigma(\epsilon)w^p \epsilon^{\beta(p-1)} \sim v^p \epsilon^{\beta(1-p)+p}$ and $\sigma(\epsilon) \frac{w}{\epsilon} \sim v$. Consequently the limit as $\epsilon \rightarrow 0$ exists at the right-hand sides of (2.24) and (2.25), and thus by (2.23) the limit as $\epsilon \rightarrow 0$ is finite. Hence there exists $\lim_{\epsilon \rightarrow 0} \sigma(\epsilon)w'(\epsilon) =$

M . If $M \neq 0$ then $w(\delta) \sim \delta^{1-2\beta}$ which is impossible since (2.23) holds. Thus $M=0$, i.e. from (2.24) we get

$$\sigma_-(\delta)w'(\delta) = \int_0^\delta \sigma_-(w^p s^{\beta(p-1)} + \beta \frac{N-1}{(R-s)s} w) ds. \quad (2.26)$$

Hence $w' > 0$ which implies, since w is nonnegative, that w is bounded as $\epsilon \rightarrow 0$. From (2.25) we get

$$\sigma_+(\delta)w'(\delta) = \int_0^\delta \sigma_+(w^p s^{\beta(p-1)} - \beta \frac{N-1}{(r_0+s)s} w) ds. \quad (2.27)$$

We now integrate $w'(s)$ in the interval $[\epsilon, \delta]$.

If $\beta < 1$ it follows from (2.23) that the integral converges for $\epsilon \rightarrow 0$, hence $w(0)$ exists and is finite. The case $\beta = 1$ is excluded by our assumption $\mu \neq 0$.

If $\beta > 1$, we insert (2.23) in (2.27) and neglect the positive term in the integral. Then

$$w'(\delta) \geq -c\delta^{-2\beta} \int_0^\delta s^{2\beta-1+1-2\beta} v(s) ds = -c\delta^{-2\beta} \int_0^\delta v(s) ds,$$

where $v(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Integration from δ to $\delta_0 < 1$ yields

$$w(\delta_0) - w(\delta) \geq \frac{c}{2\beta-2} (\delta_0^{2-2\beta} - \delta^{2-2\beta}).$$

Thus since $\beta > 1$, $w(\delta) \leq \delta^{2-2\beta} c_1$ for some positive constant c_1 . Iterating this procedure of estimating w' from below and w from above, in a finite number of steps we get that w is bounded. This completes the proof. \square

Remark 2.1 If $-\mu^* < \mu < \frac{1}{4}$ then $\beta_+ < \frac{2}{1-p}$ and by Lemma 2.1 there is a local solution which behaves like $c'\delta^{2/(1-p)}$ for a suitable $c' > 0$. This solution is smaller than δ^{β_+} and therefore $w(0) = 0$ or equivalently $u(\delta)/\delta^{\beta_+} \rightarrow 0$ as $\delta \rightarrow 0$.

However if $\mu < -\mu^*$ or equivalently $\beta_+ > \frac{2}{1-p}$, then no solution behaving like $c'\delta^{2/(1-p)}$ can exist. In fact if such a solution exists then it satisfies the assumptions of Lemma 2.4 and consequently $u \leq c\delta^{\beta_+}$. Hence

$$0 < c' = \lim_{\delta \rightarrow 0} \frac{u}{\delta^{2/(1-p)}} \leq \lim_{\delta \rightarrow 0} c\delta^{\beta_+-2/(1-p)} = 0.$$

This is impossible.

If $\lim_{\delta \rightarrow 0} \frac{u}{\delta^{\beta_+}} = w(0) \neq 0$ then the arguments developed in Proposition 2.1 for the function v apply also to the solution $w = u/\delta^{\beta_+}$.

Proposition 2.2 Assume $-\mu^* < \mu < \frac{1}{4}$ and $\frac{u(\delta)}{\delta^{\beta_-}} \rightarrow 0$ as $\delta \rightarrow 0$. Then $w(\delta) := \frac{u(\delta)}{\delta^{\beta_+}}$ has a limit $w(0)$ for $\delta \rightarrow 0$ and if $w(0) \neq 0$ we get

$$w(\delta) = w(0) + O(\delta^\alpha), \text{ where } \alpha = \min\{1, 2 - \beta_+(1 - p)\} > 0.$$

Moreover if $-\frac{p}{(1-p)^2} < \mu$, we have $\alpha = 1$ and

$$w'(0) = \frac{N-1}{2R}w(0) \text{ at the outer boundary } r = R,$$

$$w'(0) = -\frac{N-1}{2r_0}w(0) \text{ at the inner boundary } r = r_0.$$

If $w(0) = 0$ we have $w(\delta) = o(\delta^\alpha)$.

Proof. We indicate the proof for the outer boundary. The statement for the inner boundary is proved in exactly the same way.

By the arguments given in the proof of Lemma 2.4 the function w satisfies (2.26). If $-\frac{p}{(1-p)^2} \leq \mu$, i.e. $\alpha = 1$ we divide (2.26) by σ_- and apply Bernoulli l'Hospital's rule to get $w'(0)$. For $\mu = -\frac{p}{(1-p)^2}$ the derivative involves an additional term. For $-\mu^* < \mu < -\frac{p}{(1-p)^2}$, that is $0 < \alpha = 2 - \beta_+(1 - p) < 1$, we divide (2.26) by $\sigma(\delta)$ and integrate in $[0, \delta]$ to get $w(\delta) - w(0)$ (cf. (2.28)). We easily see that the second term in the integral is higher order than the first one which is of order α . By a simple analysis of it we get the conclusion

$$\lim_{\delta \rightarrow 0} \frac{w(\delta) - w(0)}{\delta^\alpha} = \frac{w^p(0)}{(2 - \beta_+(1 - p))(1 + \beta_+(1 + p))}.$$

□

The same type of argument as in Lemma 2.4 shows that there are no solutions which lie strictly between $c_1\delta^{\beta_+}$ and $c'\delta^{2/(1-p)}$.

Proposition 2.3 Assume $-\mu^* < \mu < 1/4$ or equivalently $\beta_+ < \frac{2}{1-p}$. Then no solution exists for which $\lim_{\delta \rightarrow 0} u(\delta)\delta^{\frac{2}{p-1}} = \infty$ and $\lim_{\delta \rightarrow 0} u(\delta)\delta^{-\beta_+} = 0$.

Proof. By Proposition 2.2 we have $u(\delta)\delta^{-\beta_+} = o(\delta^\alpha)$ where $\alpha = \min\{1, 2 - \beta_+(1 - p)\} > 0$. Moreover $\beta_+ < \beta_+ + \alpha \leq \beta_+ + 2 - \beta_+(1 - p) = 2 + p\beta < \frac{2}{1-p}$ by our assumption on β_+ . Let ϵ be such that

$$\beta_+(\epsilon) = 1/2 + \sqrt{1/4 - \mu + \epsilon} = \beta_+ + \alpha < \frac{2}{1-p}.$$

By $\lim_{\delta \rightarrow 0} u(\delta)\delta^{\frac{2}{p-1}} = \infty$, there exists $\delta_0 > 0$ such that $u(\delta) \geq \delta^{\frac{2}{1-p}}/\epsilon$ in $(0, \delta_0]$ and $C_H(\mathcal{A}(R - 2\delta_0, R)) = \frac{1}{4} = C_H(\mathcal{A}(r_0, r_0 + 2\delta_0))$ (cf. Corollary 2.2). Then

$$-\frac{\mu}{\delta^2}u \leq \Delta u = u(u^{p-1} - \frac{\mu}{\delta^2}) \leq u\frac{\epsilon - \mu}{\delta^2} \text{ for } \delta > \delta_0.$$

Let h be a "small" harmonic satisfying $\Delta h + \frac{\mu - \epsilon}{\delta^2} h = 0$ in a small neighborhood of the boundary. It behaves for δ near zero like $\delta^{\beta_+(\epsilon)}$. Since h is defined up to a multiplicative constant, we can always assume that $h(\delta_0) = u(\delta_0)$. Remark that h and u are in $W_0^{1,2}$ in an neighborhood of the boundary, then the comparison principle applies and yields $u \geq h$. By the choice $\beta_+(\epsilon) = \beta_+ + \alpha$ we have $u(\delta) \geq c\delta^{\beta_+ + \alpha}$ for some positive c , which contradicts the result $u(\delta)\delta^{-\beta_+} = o(\delta^\alpha)$. Then the conclusion follows. \square

2.3 Existence of local solutions at the boundary

In this section we construct local solutions at the boundary points using the results of the previous sections. If $\mu > -\mu^*$, we know that a unique solution exists which grows at the boundary like $\delta^{\frac{2}{1-p}}$ (cf. Lemma 2.1, (ii)), i.e. the non linear term is leading. Under the same condition on μ , we also prove that there exist solutions which grow at the boundary like the harmonics. It turns out that $\mu = -\frac{2(p+1)}{(p-1)^2}$ is a critical value.

Lemma 2.5 *Suppose that $\beta = \beta_+$ and $-\mu^* < \mu < \frac{1}{4}$, $\mu \neq 0$. For any positive constant $w(0)$ there exist near $r = R$ or $r = r_0$ a unique local solution of the form $u(\delta) = \delta^{\beta_+} w(\delta)$ where $w(\cdot)$ is continuous in $[0, \delta_0]$ (for some $\delta_0 \in (0, \frac{R-r_0}{2})$) and*

$$w(\delta) = w(0) + O(\delta^\alpha), \quad \text{where } \alpha = \min\{1, 2 - \beta_+(1-p)\} > 0.$$

Moreover $w(\cdot)$ behaves as described in Proposition 2.2.

Proof. Let us first consider the case $r = R$. We write $u = \delta^{\beta_+} w$ and observe that w satisfies equation (2.24) for $\beta = \beta_+$. We shall study the initial value problem (2.24) with $w(0) = w_0$ and $\lim_{\epsilon \rightarrow 0} \sigma_-(\epsilon)w'(\epsilon) = 0$. It can be transformed into the integral equation

$$w(\delta) = w(0) + \underbrace{\int_0^\delta \sigma \left(s^{\beta(p-1)} w^p + \beta \frac{N-1}{(R-s)s} w \right) \left(\int_s^\delta \sigma^{-1} d\xi \right) ds}_{A(w)}, \quad (2.28)$$

where $\sigma(s) = \sigma_-(s) = s^{2\beta}(R-s)^{N-1}$. Because of our assumption on μ , we have $\beta_+(p-1) + 1 > -1$. Hence the integral exists for finite w . Straightforward computation shows that $A(w)$ is a contraction for small δ . Hence there exists a fixed point w . Its behavior at zero follows from Proposition 2.2. The same argument applies to the inner boundary. \square

Remark 2.2 The hypothesis $\mu > -\mu^*$ is necessary for the existence of solutions of order δ^{β_+} . Indeed the opposite condition $\mu \leq -\mu^*$ is equivalent to $\beta_+(1-p) \geq 2$. Thus (2.1) cannot be satisfied since for $u(r) = w(\delta)\delta^{\beta_+}$, $w(0) > 0$, its left hand side depends on δ with order higher than $\beta_+ - 2$ and its right hand side is of order $p\beta_+ \leq \beta_+ - 2$.

Proposition 2.4 Assume $\mu < -\frac{2(p+1)}{(1-p)^2}$ and $\lim_{\delta \rightarrow 0} \frac{u(\delta)}{\delta^{\beta_-}} = 0$. Then $u(x) \equiv 0$ in a neighborhood of the boundary.

Proof. By contradiction suppose that such a local solution is positive in $[0, \delta_0)$ ($\delta_0 > 0$).

By lemma 2.4 our hypothesis on u implies

$$u(\delta) \leq c\delta^{\beta_+}, \quad \delta \in [0, \delta_0]. \quad (2.29)$$

For any $\epsilon \in [0, \delta_0)$ we define z_ϵ as in the proof of Corollary 2.2 for a constant $C = C_0$ such that z_ϵ is an upper solution of (1.1). The hypothesis on μ gives $\beta_+ > \frac{2}{1-p}$, hence by (2.29) for a possibly smaller δ_0 we get

$$u(\delta) \leq z_0(\delta), \quad \delta \in [0, \delta_0].$$

For $\epsilon \in (0, \delta_0)$ sufficiently small we have $z(\delta_0) > u(\delta_0)$. Then $\delta_1 \in (0, \delta_0)$ exists such that $z_\epsilon(\delta) < u(\delta)$ in $(0, \delta_1)$ and $z_\epsilon(\delta_1) = u(\delta_1)$. As in the proof of Corollary 2.2, this is impossible by the comparison principle. \square

In Lemma 2.5 we constructed solutions that vanish on the inner or outer boundary and belong to the space $W^{1,2}$ in a neighborhood of it. Here we prove the existence of "singular" local solutions near the boundary.

Lemma 2.6 Let $\beta = \beta_- \in (0, 1/2)$. For given $v(0) > 0$ and $C \in \mathbb{R}$ there exists near $r = R$ or $r = r_0$ a unique local solution of the following form respectively

$$u(r) = \delta^{\beta_-} v(0) \left(1 + \frac{N-1}{2R} \delta + o(\delta)\right) + C\delta^{\beta_+}, \quad \text{if } \delta = R - r, \quad (2.30)$$

$$u(r) = \delta^{\beta_-} v(0) \left(1 - \frac{N-1}{2r_0} \delta + o(\delta)\right) + C\delta^{\beta_+}, \quad \text{if } \delta = r - r_0. \quad (2.31)$$

Proof. We look for a solution of the form $u = \delta^{\beta_-} v$ where v satisfies (2.16). At the outer boundary it can be written in view of Proposition 2.1 as an integral equation of the form

$$\begin{aligned} v(\delta) - v(0) - L \int_0^\delta \sigma_-^{-1} ds = \\ \int_0^\delta \sigma_- \left(v^p s^{\beta_-(p-1)} + \beta_- \frac{N-1}{(R-s)s} v \right) \int_s^\delta \sigma_-^{-1} d\xi ds, \end{aligned}$$

where $\sigma_-(s) = s^{2\beta_-}(R-s)^{N-1}$ and $L = CR^{N-1}(1-2\beta_-)$. We can write $v(\delta) = v(0) + L \int_0^\delta \sigma_-^{-1} ds + \delta\eta(\delta)$ and use a standard fixed point theorem to prove the existence of η . Moreover since $1-2\beta_- = \beta_+$ it follows that $\delta^{\beta_-} L \int_0^\delta \sigma_-^{-1} ds = C\delta^{\beta_+} + \delta^{\beta_-} o(\delta)$. Likewise we establish a solution at the inner boundary. \square

The remaining case $\beta_- < 0$ which requires a more subtle argument, is covered in the next lemma.

Lemma 2.7 *Assume $\beta = \beta_- < 0$. Let $v(0)$ be an arbitrary positive constant. Then there exists near $r = R$ or $r = r_0$ a one parameter family of local solutions of the form*

$$u(r) = \delta^{\beta_-} v(0) \left(1 + \frac{N-1}{2R} \delta + \eta(\delta, C)\right) + C\delta^{\beta_+}$$

or

$$u(r) = \delta^{\beta_-} v(0) \left(1 - \frac{N-1}{2r_0} \delta + \eta(\delta, C)\right) + C\delta^{\beta_+},$$

respectively, where

$$\eta(\delta, C) = \begin{cases} o(\delta^{1-2\beta_-}) & \text{if } -\frac{1}{2} < \beta_- < 0, \\ K\delta^2 |\log(\delta)| + o(\delta^2 |\log(\delta)|) & \text{if } \beta_- = -\frac{1}{2}, \\ K\delta^2 + o(\delta^2) & \text{if } \beta_- < -\frac{1}{2}. \end{cases}$$

where K is a constant which depends on the data of the problem but not on $v(0)$ neither on the parameter $C > 0$ while the higher order terms do depend on both $v(0)$ and $C > 0$. If $\beta \leq -\frac{1}{2}$ the term $C\delta^{\beta_+}$ would be included in the higher order term but we wrote it explicitly since this is the parameter on which solutions depend.

Proof. As before we carry out the proof only for the outer boundary. Equation (2.17) can be written as

$$\frac{(v'(R-\delta)^{N-1})'}{(R-\delta)^{N-1}} + \frac{2}{\delta} \beta v' - \beta \underbrace{\frac{N-1}{(R-\delta)\delta}}_{\frac{N-1}{R}[\frac{1}{\delta} + \frac{1}{R-\delta}]} v = v^p \delta^{\beta(p-1)}.$$

If we integrate this expression we get

$$\begin{aligned} v'(\delta)(R-\delta)^{N-1} - v'(0)R^{N-1} + \beta \int_0^\delta (R-s)^{N-1} \left\{ 2\frac{v'}{s} - \frac{N-1}{R} \frac{v}{s} \right\} ds \\ = \int_0^\delta (R-s)^{N-1} \{ v^p s^{\beta(p-1)} + \beta \frac{N-1}{R(R-s)} v \} ds. \end{aligned} \quad (2.32)$$

Notice that the integral at the left is singular without additional conditions on v and v' . Set

$$v(\delta) = v(0) \left(1 + \frac{N-1}{2R} \delta + \eta(\delta) \right), \quad v'(\delta) = v(0) \left(\frac{N-1}{2R} + \eta' \right).$$

The behavior of the function η near the origin will be specified later. For the moment we assume that all the integrals which appear in the calculations below are well-defined. Substituting v and v' we get for the different expressions in the equation (2.32)

$$\begin{aligned} v'(\delta)(R-\delta)^{N-1} - v'(0)R^{N-1} &= v(0)\eta'(R-\delta)^{N-1} \\ &\quad - v(0)\frac{(N-1)^2}{2}R^{N-3}\delta - \eta_1, \end{aligned} \quad (2.33)$$

$$-\beta \int_0^\delta (R-s)^{N-1} \left\{ 2\frac{v'}{s} - \frac{N-1}{R} \frac{v}{s} \right\} ds = -2\beta v(0) \int_0^\delta (R-s)^{N-1} \frac{\eta'}{s} ds \quad (2.34)$$

$$+ \beta v(0) \frac{(N-1)^2}{2} R^{N-3} \delta + \beta v(0) \frac{N-1}{R} \int_0^\delta (R-s)^{N-1} \frac{\eta}{s} ds + \eta_2$$

$$\int_0^\delta (R-s)^{N-1} \left\{ v^p s^{\beta(p-1)} + \beta \frac{N-1}{R(R-s)} v \right\} ds = \int_0^\delta (R-s)^{N-1} v^p s^{\beta(p-1)} ds \quad (2.35)$$

$$+ \beta v(0)(N-1)R^{N-3}\delta + v(0) \frac{\beta(N-1)}{R} \int_0^\delta (R-s)^{N-2} \eta ds + \eta_3.$$

Here the functions η_i , $i = 1, 2, 3$, are of order $O(\delta^2)$ and are independent of η and η' . In the sequel we shall use the following notation:

$$\begin{aligned} A &:= v(0)R^{N-3} \left\{ \frac{(N-1)^2}{2} + \beta(N-1) + \frac{\beta(N-1)^2}{2} \right\}, \\ y_1 &:= v(0)(R-\delta)^{N-1}\eta' \text{ and } y_2 := v(0)(R-\delta)^{N-1}\eta, \\ H(\delta, y_2) &:= \int_0^\delta (R-s)^{N-1} v^p s^{\beta(p-1)} ds + \beta \frac{N-1}{R} \int_0^\delta \left(y_2 + \frac{y_2}{s} \right) ds + \sum_1^3 \eta_i, \end{aligned}$$

where $v(s)$ is replaced by $v(0)(1 + \frac{N-1}{2R}s + \frac{y_2(s)}{v(0)(R-s)^{N-1}})$. For the next arguments it will be important to keep in mind that H is independent of y_1 . From (2.33), (2.34) and (2.35) it follows that

$$y_1(\delta) = -2\beta \int_0^\delta \frac{y_1}{s} ds + A\delta + H(\delta, y_2). \quad (2.36)$$

Moreover

$$y_2(\delta) = (R - \delta)^{N-1} \int_0^\delta \frac{y_1(s)}{(R - s)^{N-1}} ds. \quad (2.37)$$

If we set $\phi(\delta) := \int_0^\delta \frac{y_1}{s} ds$ then (2.36) can be written as

$$\phi'(\delta)\delta = -2\beta\phi(\delta) + A\delta + H(\delta, y_2), \quad (2.38)$$

and solved by the variation - of - constants formula, as we will do below. Clearly

$$y_1 = -2\beta\phi + A\delta + H(\delta, y_2) =: T(y_2). \quad (2.39)$$

We now replace y_1 in (2.37) by $T(y_2)$ and we obtain a fixed point equation for y_2 , namely

$$y_2(\delta) = (R - \delta)^{N-1} \int_0^\delta \frac{T(y_2)}{(R - s)^{N-1}} ds =: \Theta(y_2).$$

Next we want to show that in a properly chosen set, $\Theta(w)$ is a contraction. For this purpose we distinguish between three cases.

(i) $-\frac{1}{2} < \beta < 0$.

Consider the Banach space $X := \{w \in C([0, \delta_0]) : |w| \leq M\delta^{-2\beta+1}, \delta \in [0, \delta_0]\}$ where M and $\delta_0 \leq \frac{R-r_0}{2}$ are positive constants which will be determined later, and $\|w\| := \sup\{|\frac{w_2}{\delta^{-2\beta+1}}|, \delta \in (0, \delta_0]\}$. By the variation-of-constants formula (2.38) gives

$$\phi = c\delta^{-2\beta} + \frac{A}{2\beta+1}\delta + \delta^{-2\beta} \int_0^\delta s^{2\beta-1} H(s, w) ds$$

where c is an arbitrary parameter.

First we estimate $|T(w) - T(\tilde{w})| = |2\beta(\tilde{\phi} - \phi) + H(\delta, w) - H(\delta, \tilde{w})|$ for w and \tilde{w} in X .

Set for short $v(\delta) = (R - \delta)^{1-N}(\gamma + w)$ where $\gamma = (R - \delta)^{N-1}v(0)(1 + \frac{N-1}{2R}\delta)$ and similarly $\tilde{v}(\delta)$ with w replaced by \tilde{w} . For fixed M we can take a sufficiently small δ_0 such that $\gamma + w \geq (R - \delta)^{N-1}v(0)$.

Indeed $\delta_0 \leq \frac{R-r_0}{2}$ implies $\gamma - (R - \delta)^{N-1}v(0) \geq \frac{(R+r_0)^{N-1}}{2^N R} v(0)\delta = c_0\delta \geq M\delta^2 \geq |w|$ if

$$\delta_0 \leq \frac{c_0}{M} \quad (2.40)$$

Then the following inequality holds

$$|v^p - \tilde{v}^p| \leq \frac{p}{(R - \delta)^{(N-1)p} v(0)^{1-p}} |w - \tilde{w}|. \quad (2.41)$$

Then

$$\begin{aligned} \left| \int_0^\delta (R-s)^{N-1} s^{\beta(p-1)} (v^p - \tilde{v}^p) ds \right| &\leq c_1 \delta^{\beta(p-3)+2} \|w - \tilde{w}\|, \\ |H(\delta, w) - H(\delta, \tilde{w})| &\leq c_1 \delta^{\beta(p-3)+2} \|w - \tilde{w}\| + c_2 \delta^{-2\beta+2} \|w - \tilde{w}\| \\ &\quad + c_3 \delta^{-2\beta+1} \|w - \tilde{w}\| \leq c_4 \delta^{-2\beta+1} \|w - \tilde{w}\|, \end{aligned}$$

where $c_i > 0$ ($i \in \mathbb{N}$) stand for constants independent of δ all along this proof. Furthermore

$$\begin{aligned} |\phi(\delta) - \tilde{\phi}(\delta)| &= |\delta^{-2\beta} \int_0^\delta s^{2\beta-1} (H(s, \tilde{w}) - H(s, w)) ds| \\ &\leq c_4 \delta^{-2\beta+1} \|w - \tilde{w}\|. \end{aligned}$$

Therefore

$$|T(w) - T(\tilde{w})| \leq (-2\beta c_4 + c_4) \delta^{-2\beta+1} \|w - \tilde{w}\| = c_4 \delta^{-2\beta+1} \|w - \tilde{w}\|.$$

Hence

$$|\Theta(w) - \Theta(\tilde{w})| \leq \frac{c_5}{-2\beta+2} \delta^{-2\beta+2} \|w - \tilde{w}\|$$

and

$$\|\Theta(w) - \Theta(\tilde{w})\| \leq c_6 \delta_0 \|w - \tilde{w}\|.$$

For given M , δ_0 can be chosen possibly smaller so that $\Theta(w)$ is a contraction and (2.40) holds. It remains to show that $\Theta : X \rightarrow X$. For $w \in X$ we have by (2.39)

$$|T(w)| \leq |2\beta c| \delta^{-2\beta} + \frac{|A|}{2\beta+1} \delta + c_1 M \delta^{-2\beta+1},$$

where c_1 is independent of M and δ_0 . Consequently

$$\begin{aligned} |\Theta(w)| &\leq \frac{2|\beta c|}{-2\beta+1} \delta^{-2\beta+1} + \frac{|A|}{2(2\beta+1)} \delta^2 + c_2 \delta^{-2\beta+2}, \\ \|\Theta(w)\| &\leq \frac{2|\beta c|}{-2\beta+1} + \frac{|A|}{2(2\beta+1)} \delta^{1+2\beta} + c_2 \delta. \end{aligned}$$

We now fix $M > -2\beta c$ and choose δ_0 sufficiently small such that $\|\Theta(w)\| \leq M$. Notice that by decreasing δ_0 the inequality (2.40) is not violated. Then $\Theta(w)$ is a contraction in X and the conclusion follows. Indeed $C = \frac{-2\beta}{(1-2\beta)R^{N-1}} c \geq 0$ follows from the representation formula of the solution, as a fixed point, and from $\beta_+ = \beta_- + (1-2\beta_-)$. This completes the proof for $\beta \in (-1/2, 0)$

If $c = 0$ we can carry out the same proof in the space $X := \{w \in C([0, \delta_0]) : |w| \leq M\delta^2, \delta \in [0, \delta_0]\}$ with the norm $\|w\| := \sup\{|\frac{w^2}{\delta^2}|, \delta \in$

$(0, \delta_0]$. Here M will be a constant close to $\frac{|A|}{2\beta+1}$ which is the leading term in $\|\Theta(w)\|$ if $A \neq 0$. For $c = 0$ the solution is C^2 up to the boundary.

(ii) $\beta = -\frac{1}{2}$.

In this case we have

$$\phi = c\delta + A\delta|\log \delta| + \delta \int_0^\delta H s^{-2} ds.$$

Here $-\beta + 1 = 2$ and we argue exactly as before if $A = 0$, i.e. $N = 1$ or $N = 3$. Otherwise the logarithmic term prevails. We then take $|w| \leq M\delta^2 \log(1/\delta)$ and $\|w\| := \sup\{|\frac{w_2(\delta)}{\delta^2 \log \delta}|, \delta \in (0, \delta_0]\}$. It turns out that for small δ_0 , $\Theta(w)$ is a contraction which maps the ball $\{|w| \leq M\delta^2 \log \delta\}$ into itself. It has therefore a fixed point. The details will be omitted.

(iii) $\beta < -\frac{1}{2}$.

The function ϕ defined before is in general not defined for $\delta = 0$ unless we impose strong growth conditions on w at zero. We therefore express the solution of (2.36) by means of the modified function

$$\phi(\delta) = \int_0^\delta \frac{y_1}{s} ds \quad (2.42)$$

$$= c\delta^{-2\beta} + \frac{A\delta}{2\beta+1} - \delta^{-2\beta} \int_\delta^{\delta_0} H s^{2\beta-1} ds. \quad (2.43)$$

In this case the leading term of ϕ is of order $O(\delta)$ provided $A \neq 0$. If $A = 0$ it is of higher order. We consider the operator $\Theta(w)$ in the Banach space $X := \{w \in C([0, \delta_0]) : |w| \leq M\delta^2, \delta \in [0, \delta_0]\}$, where M and $\delta_0 \leq \frac{R-r_0}{2}$ are positive constants which will be determined later, and $\|w\| := \sup\{|\frac{w(\delta)}{\delta^2}|, \delta \in (0, \delta_0]\}$. The estimates are similar to the ones in the first case except that

$$\begin{aligned} |H(\delta, w) - H(\delta, \tilde{w})| &\leq c_4 \delta^2 \|w - \tilde{w}\|, \\ |\phi(\delta) - \tilde{\phi}(\delta)| &\leq c_5 \delta^2 \|w - \tilde{w}\|. \end{aligned}$$

If $\beta \neq -1$, as before this leads to $\|\theta(w) - \Theta(\tilde{w})\| \leq c_6 \delta_0 \|w - \tilde{w}\|$. Notice that if $\beta \neq -1$ the expression $\delta^{-2\beta} \int_\delta^{\delta_0} H s^{2\beta-1} ds$ is of order $O(\delta^2)$. For the next claim that $\Theta : X \rightarrow X$ we observe that for $w \in X$

$$|T(w)| \leq c\delta^{-2\beta} + \frac{|A|}{|2\beta+1|} \delta + c_2 M \delta^2.$$

Then

$$\|\Theta(w)\| \leq c\delta_0^{-2\beta-1} + \frac{|A|}{2|2\beta+1|} + c_2 M \delta_0 < M,$$

for $M > \frac{|A|}{2|2\beta+1|}$ and δ_0 sufficiently small. From here we conclude that $\Theta : X \rightarrow X$ is a contraction and has a unique fixed point.

If $\beta = -1$, then $\delta^{-2\beta} \int_{\delta}^{\delta_0} H s^{2\beta-1} ds$ is of order $O(\delta^2 |\ln \delta|)$. By requiring that $\delta_0 |\ln \delta_0|$ is sufficiently small we obtain that Θ is a contraction in X .

Notice that the dependence of the constant C from c is not explicit in this case.

□

Remark 2.3 *The constant A vanishes if $\beta = -\frac{N-1}{N+1}$ or if $N = 1$. If both c and A vanish higher order terms come into play. The discussion is straightforward and will be omitted.*

3 Global solutions

3.1 Ball

Theorem 3.1 *Assume $\mu < 1/4$, $\mu \neq 0$. For $\Omega = B_R$ we have*

(i) *For any given $u(0) > 0$ problem (1.1) possesses in the ball a unique positive radial solution. At the boundary it behaves like $c\delta^{\beta-}$, for some $c > 0$. The solutions are monotone increasing with respect to $u(0)$.*

(ii) *For any $0 < R_0 < R$ there exists a nonnegative radial solution in the ball with a dead core in B_{R_0} . At the boundary it behaves like $c\delta^{\beta-}$ for some $c > 0$.*

(iii) *There exists a solution of the form $u(r) = r^{\frac{2}{1-p}}(c'' + w(r))$ with $c'' = (-\mu^* + \frac{2(N-1)}{1-p})^{\frac{1}{p-1}}$ and $w(0) = 0$. At the boundary it behaves like $c\delta^{\beta-}$, for some $c > 0$.*

Proof. From Section 2.1 we know that problem (2.1) with the initial conditions $u(0) = u_0 > 0$ and $u'(0) = 0$ has a unique local solution which can be continued until it vanishes or it blows up. Since $p < 1$ blow up cannot occur for $r < R$. If $\mu < 1/4$ then by the comparison principle stated in the introduction it cannot vanish before $r = R$. By the results of the previous section it behaves at the boundary like $c\delta^{\beta-}$ with $c > 0$ or it is bounded from above by $c\delta^{\beta+}$. The second case is impossible in view of the comparison principle. Consequently $u \sim c\delta^{\beta-}$ at the boundary. Solutions are monotone increasing with respect

to $u(0)$ since they cannot intersect for $r \in (0, R)$ by the comparison principle.

These solutions are positive in the whole ball. All other solutions have a dead core. In fact if we choose $R_0 > 0$, set $u = 0$ in $[0, R_0]$ and continue it with the solution constructed in Section 2.1, by the same arguments as before we obtain a solution which exists in the whole ball and behaves at the boundary like $c\delta^{\beta-}$. Notice that a solution for which $u(R_0) = 0$ and $u'(R_0) = 0$ is necessarily zero in $(0, R_0)$.

The third assertion follows from Lemma 2.1 (iii). \square

If $\mu < 0$ the solutions are monotone increasing and blow up at the boundary. This is not the case if $\mu > 0$.

Notice that the solution with a dead core at the boundary has a singularity at the origin.

3.2 Annulus

The structure of the positive *radial solutions* in an annulus is described in

Theorem 3.2 *For $\Omega = \mathcal{A}(r_0, R)$ and $\mu < C_H(\mathcal{A}(r_0, R))$, $\mu \neq 0$, we have*

(i) *For any given $r_0 < R_0 < R$ there exists a unique solution positive in (R_0, R) , with a dead core in $[r_0, R_0]$. At the outer boundary it behaves like $k(R - r)^{\beta-}$, for some $k > 0$. Vice versa*

For any given $r_0 < R_0 < R$ there exists a unique solution positive in (r_0, R_0) , with a dead core in $[R_0, R]$. At the inner boundary it behaves like $k(r - r_0)^{\beta-}$, for some $k > 0$.

(ii) *The sum of two solutions as in (i), having a disjoint support, is a solution with a dead core interval (eventually reduced to a point) and positive near the inner and outer boundary.*

(iii) *If $\mu < 0$, for any given $r_0 < R_0 < R$ and $u(R_0) = u_0 > 0$ there exists a unique positive solution. At the outer and inner boundaries it behaves like $k_1(R - r)^{\beta-}$, respectively $k_2(r - r_0)^{\beta-}$, for some $k_1, k_2 > 0$.*

If moreover $\mu > -\frac{2(p+1)}{(1-p)^2}$ we have

(iv) *For any given $c > 0$ there exists a unique positive solution such that $u(r)/(r - r_0)^{\beta+} \rightarrow c$ as $r \rightarrow r_0$. At the outer boundary it behaves like $k(R - r)^{\beta-}$, for some $k > 0$. Vice versa we have*

for any given $c > 0$ there exists a unique positive solution such that $u(r)/(R - r)^{\beta+} \rightarrow c$ as $r \rightarrow R$. At the inner boundary it behaves like $k(r - r_0)^{\beta-}$, for some $k > 0$.

(v) *There exists a unique solution such that $u(r)/(r - r_0)^{\frac{2}{1-p}} \rightarrow c'$ as $r \rightarrow r_0$. At the outer boundary it behaves like $k(R - r)^{\beta-}$, for some $k > 0$. Here $c' := \left(\frac{2(1+p)}{(p-1)^2} + \mu \right)^{1/(p-1)}$.*

The same holds if we interchange the role of the inner and outer boundary.

Proof. In order to prove the first statement let \tilde{u} be the solution with a dead core in one point $r = R_0$ constructed in Lemma 2.1. The same arguments that we used in the proof of Theorem 3.1 give that \tilde{u} can be continued to the right and the left until it reaches the inner and outer boundary. There it behaves like $k\delta^{\beta-}$, where δ denotes the distance from the boundary and $k > 0$. If in $[r_0, R_0]$ (respectively in $[R_0, R]$) we replace it with $\tilde{u} \equiv 0$, we get (i).

(ii) is a simple remark.

(iii) As already remarked, problem (2.1), (2.3) has a local solution. Moreover for $u_1 = 0$ and $\mu < 0$ this solution increases in $[R_0, R]$ and decreases in $(r_0, R_0]$, hence it is positive and cannot go to 0 at the boundary. By Corollary 2.3 the solution behaves as $k\delta^{\beta-}$ at the inner and outer boundary. By Lemma 2.4 we have $k > 0$.

(iv) we start with the local solution which behaves at the inner or outer boundary like $c\delta^{\beta+}$ (see Lemma 2.5). It can be continued till the outer or inner boundary. Then we argue as in Theorem 3.1.

(v) is proved exactly on the same line (see Lemma 2.1, (ii)). \square

3.3 General domains

In this section we shall construct solutions of (1.1) in arbitrary not necessarily simply connected domains. More precisely we shall prove the following theorem.

Theorem 3.3 *Let $\mu < \frac{1}{4}$, $\mu \neq 0$, and Ω be a bounded domain with C^k ($k \geq 2$) boundary. Then the following statements hold for the solutions of (1.1):*

- (i) *for suitable $0 < c_0 < c_1$ there exists a solution u such that $0 < c_0 \leq u(x)/\delta^{\beta-}(x) \leq c_1$ in a neighborhood of $\partial\Omega$.*
- (ii) *If c_0 and c_1 are sufficiently small this solution has a dead core in the interior of Ω .*
- (iii) *If $\partial\Omega$ is not connected, then for any non empty, closed, disjoint sets Γ_1, Γ_2 , such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, and for suitable sufficiently small $0 < c_0 < c_1$, there exists a solution u positive in a neighborhood of Γ_1 where it behaves as in (i) and such that $u(x) \equiv 0$, in a neighborhood of Γ_2 .*

For the proof of the theorem we need some properties of the distance function $\delta(x)$ where x is an arbitrary point in Ω . Denote by Ω_ρ the parallel set $\{x \in \Omega : \delta(x) < \rho\}$. If Ω is of class C^k , $k \geq 2$, then δ

is in $C^k(\Omega_{\rho_0})$ for $\rho_0 > 0$ sufficiently small. Denote by $\sigma(x)$ the nearest point to x on $\partial\Omega$. Let $K_i(\sigma(x))$, $i = 1, \dots, N-1$ be the principal curvatures and $H(\sigma(x)) = \sum_{i=1}^{N-1} \frac{K_i}{N-1}$ be the mean curvature. Then for any $x \in \Omega_{\rho_0}$

$$\begin{aligned} |\nabla\delta(x)| &= 1, \\ -\frac{N-1}{\rho_0 - \delta(x)} &\leq \Delta\delta(x) = -\sum_{i=1}^{N-1} \frac{K_i}{1 - K_i\delta(x)} \leq \frac{N-1}{\rho_0 + \delta(x)}. \end{aligned} \quad (3.1)$$

Proof of Theorem 3.3.

(i) For the proof of the first assertion we shall distinguish between two cases.

(A) $\mu \in (0, 1/4)$.

For $0 < s \leq \rho < \frac{\rho_0}{2}$, $\epsilon > 0$, let $\phi(s) := Ms^{\beta_-}(\rho^\epsilon - s^\epsilon)$. Then

$$\begin{aligned} \phi'(s) &= \beta_- Ms^{\beta_- - 1}(\rho^\epsilon - \frac{\beta_- + \epsilon}{\beta_-} s^\epsilon), \\ \phi''(s) &:= \beta_-(\beta_- - 1) \frac{\phi(s)}{s^2} - M\epsilon(2\beta_- + \epsilon - 1)s^{\beta_- + \epsilon - 2}. \end{aligned}$$

The function $\tilde{u}(x) := \phi(\delta(x))$ is well defined for $x \in \Omega_{\rho_0}$, and it satisfies (by (3.1)):

$$\begin{aligned} \Delta\tilde{u}(x) &= \phi''(\delta)|\nabla\delta|^2 + \phi'(\delta)\Delta\delta \\ &= -\mu \frac{\phi(\delta)}{\delta^2} - M\epsilon(2\beta_- + \epsilon - 1)\delta^{\beta_- + \epsilon - 2} + \phi'(\delta)\Delta\delta. \end{aligned} \quad (3.2)$$

By a suitable choice of ϵ we can construct *local upper and lower solutions*. In fact:

- (a) if $0 < 1 - 2\beta_- < \epsilon < 1$, then there exists $\rho < \frac{\rho_0}{2}$ sufficiently small such that, for any $M > 0$, \tilde{u} is an upper solution in Ω_ρ .
- (b) For any given $0 < \epsilon := \underline{\epsilon} < 1 - 2\beta_- < 1$ and $M > 0$ there exists $\rho < \frac{\rho_0}{2}$ sufficiently small such that \tilde{u} is a lower solution in Ω_ρ .

The first assertion (a) follows from the estimate

$$|\phi'(s)| \leq \beta_- Ms^{\beta_- - 1} \rho^\epsilon \max\{1, \frac{\epsilon}{\beta_-}\} \leq MKs^{\beta_- - 1}$$

for some constant K independent of s , and

$$|\Delta\delta(x)| \leq K_1 \text{ in } \Omega_\rho,$$

where K_1 depends only on ρ_0 . Inserting these estimates into (3.2) we get

$$\begin{aligned}\Delta \tilde{u} + \mu \frac{\tilde{u}}{\delta^2} - \tilde{u}^p &\leq -\epsilon(2\beta_- + \epsilon - 1)M\delta^{\beta_- + \epsilon - 2} + MKK_1\delta^{\beta_- - 1} \\ &= -M\delta^{\beta_- + \epsilon - 2} [\epsilon(\epsilon - (1 - 2\beta_-)) - KK_1\delta^{1-\epsilon}].\end{aligned}$$

For small δ the right-hand side is negative. This proves the first assertion.

The second assertion (b) follows from

$$\begin{aligned}\Delta \tilde{u} + \mu \frac{\tilde{u}}{\delta^2} - \tilde{u}^p &\geq M[\underline{\epsilon}(1 - 2\beta_- - \underline{\epsilon})\delta^{\beta_- + \underline{\epsilon} - 2} - KK_1\delta^{\beta_- - 1} - K_2M^{p-1}\delta^{p\beta_-}] \\ &= M\delta^{\beta_- + \underline{\epsilon} - 2}[\underline{\epsilon}(1 - 2\beta_- - \underline{\epsilon}) - KK_1\delta^{1-\underline{\epsilon}} - K_2M^{p-1}\delta^{2-(1-p)\beta_- - \underline{\epsilon}}] > 0,\end{aligned}$$

where K_2 depends only on ρ_0 and ϵ . Since $2 - (1 - p)\beta_- - \epsilon > 0$ the right-hand side is positive for small ρ . This completes the proof of (b).

Next we want to extend the local upper and lower solutions constructed above to the whole domain. Let $\rho \in (0, \frac{\rho_0}{2}]$ be such that

$$\bar{u} = M\delta^{\beta_-}(\rho^\epsilon - \delta^\epsilon)$$

is an upper solution in Ω_ρ .

Observe that \bar{u} attains its maximum \bar{u}_M at $\{x \in \Omega : \delta(x) = \bar{\delta} := (\frac{\beta_-}{\beta_- + \epsilon})^{\frac{1}{\epsilon}}\rho\}$.

We choose M so small that the following inequality holds:

$$\mu \frac{\bar{u}_M}{\bar{\delta}^2} - \bar{u}_M^p = \bar{u}_M^p [\mu \frac{\bar{u}_M^{1-p}}{\bar{\delta}^2} - 1] < 0. \quad (3.3)$$

Then the constant \bar{u}_M is an upper solution of (1.1) in $\Omega \setminus \Omega_{\bar{\delta}}$ and we obtain the following (weak) global upper solution

$$\bar{U}(x) := \begin{cases} \bar{u}(x), & x \in \Omega_{\bar{\delta}}, \\ \bar{u}_M, & x \in \Omega \setminus \Omega_{\bar{\delta}}. \end{cases} \quad (3.4)$$

For the same M let $\underline{\rho} \in (0, \bar{\delta})$ be such that $\underline{u} = M\delta^{\beta_-}(\underline{\rho}^\epsilon - \delta^\epsilon)$ is a lower solution in $\Omega_{\underline{\rho}}$, such that $\bar{u} > \underline{u}$ in $\Omega_{\underline{\rho}} \subset \Omega_{\bar{\delta}}$. The function

$$\underline{U}(x) = \begin{cases} \underline{u} & \text{in } \Omega_{\underline{\rho}}, \\ 0 & \text{otherwise} \end{cases}$$

is a global lower solution.

Hence there exist an upper and a lower solution $\underline{U}(x) \leq \overline{U}(x)$ in Ω . The method of upper and lower solutions can be generalized to our case cf. [2] (Lemma 4.12) and guarantees the existence of a non trivial positive solution $\underline{U} \leq u \leq \overline{U}$.

(B) $\mu < 0$.

We start with the construction of an upper solution. Let $\bar{\sigma}$ be a positive number smaller than ρ_0 and for any given $M > 0$ let $\eta = \eta(r)$ be the solution of

$$\begin{cases} \eta'' + \frac{(N-1)}{r}\eta' + \frac{\mu}{(\rho_0-r)^2}\eta = \eta^p, & r \in (\rho_0 - \bar{\sigma}, \rho_0), \\ \eta(\rho_0 - \bar{\sigma}) = M, & \eta'(\rho_0 - \bar{\sigma}) = 0. \end{cases}$$

Since $\mu < 0$ the function $\eta(r)$ is increasing to the right and can therefore be extended as a positive solution in $(\rho_0 - \bar{\sigma}, \rho_0)$. Then

$$\lim_{r \rightarrow \rho_0} \frac{\eta(r)}{(\rho_0 - r)^{\beta_-}} = C_M > 0. \quad (3.5)$$

Indeed by Lemma 2.4 (ii), if $C_M = 0$ then $\eta(r) \rightarrow 0$ as $r \rightarrow \rho_0$ which contradicts the increasing behavior of η .

We can easily verify that the following function

$$\bar{u}(x) := \begin{cases} \eta(\rho_0 - \delta(x)), & x \in \Omega_{\bar{\sigma}}, \\ M, & x \in \Omega \setminus \Omega_{\bar{\sigma}}. \end{cases} \quad (3.6)$$

is a (weak) upper solution of (1.1). Indeed since $\mu < 0$, any constant is an upper solution. Since $\bar{u} \in C^1(\Omega)$, we only have to verify that it is a classical upper solution for any $x \in \Omega_{\bar{\sigma}}$. Indeed remark that η' is positive, hence for any $x \in \Omega_{\bar{\sigma}}$ we have by (3.1) and (3.2)

$$\begin{aligned} \Delta \bar{u}(x) &= \eta''(\rho_0 - \delta(x))|\nabla \delta|^2 - \eta'(\rho_0 - \delta(x))\Delta \delta(x) \\ &\leq \eta''(\rho_0 - \delta(x)) + \frac{(N-1)}{\rho_0 - \delta(x)}\eta'(\rho_0 - \delta(x)) \\ &= -\frac{\mu}{\delta(x)^2}\eta + \eta^p = -\frac{\mu}{\delta(x)^2}\bar{u} + \bar{u}^p. \end{aligned} \quad (3.7)$$

In order to construct a lower solution take $\underline{\sigma} \in (0, \rho_0)$ and let $z = z(d)$ be the non trivial (dead core) solution of

$$\begin{cases} z'' + \frac{(N-1)}{\rho_0+d}z' + \frac{\mu}{d^2}z = z^p, & d \in (0, \underline{\sigma}) \\ z(\underline{\sigma}) = 0, & z'(\underline{\sigma}) = 0. \end{cases}, \quad (3.8)$$

such that $z(d) > 0$, $d \in (0, \underline{\sigma})$. We extend it by 0 for $d \geq \underline{\sigma}$, set $w(r) := z(r - \rho_0)$ and observe that it is a radial solution of (1.1) in the annulus $\mathcal{A}(\rho_0, R)$, for any $R > \rho_0 + 2\underline{\sigma}$. In addition

$$\lim_{d \rightarrow 0} \frac{z(d)}{d^{\beta_-}} = C_{\underline{\sigma}} > 0. \quad (3.9)$$

We can easily verify that the following function

$$\underline{u}(x) := \begin{cases} z(\delta(x)), & x \in \Omega_{\underline{\sigma}}, \\ 0, & x \in \Omega \setminus \Omega_{\underline{\sigma}}. \end{cases} \quad (3.10)$$

is a (weak) lower solution of (1.1). Indeed $\underline{u} \in C^1(\Omega)$ and it satisfies (1.1) in the classical sense in the interior of the region where it vanishes. Hence we only have to verify that it is a classical lower solution for any $x \in \Omega_{\underline{\sigma}}$. Indeed remark that z' is negative, hence for any $x \in \Omega_{\underline{\sigma}}$, by (3.2) we have

$$\begin{aligned} \Delta \underline{u}(x) &= z''(\delta(x)) + z'(\delta(x)) \Delta \delta(x) \\ &\geq z''(\delta(x)) + \frac{(N-1)}{\rho_0 + \delta(x)} z'(\delta(x)) \\ &= -\frac{\mu}{\delta(x)^2} z + z^p = -\frac{\mu}{\delta(x)^2} \underline{u} + \underline{u}^p. \end{aligned} \quad (3.11)$$

It is not difficult to see that by choosing $\underline{\sigma}$ sufficiently small we can achieve that $\underline{u} \leq \bar{u}$. Hence the proof is complete.

(ii) We distinguish between two cases as in (i).

(A) $\mu \in (0, \frac{1}{4})$.

We construct an upper (weak) solution with dead core. Let $\bar{U}(x)$ be the upper solution in (i), **(A)**, and $\bar{\delta} \in (0, \rho_0)$ the constant used in its definition. Take $\rho \in (\bar{\delta}, \rho_0)$. By Lemma 2.1 there exists $\eta = \eta(r)$ solution of

$$\begin{cases} \eta'' + \frac{(N-1)}{r} \eta' + \frac{\mu}{(\rho_0 - r)^2} \eta = \eta^p, & r \in (\rho_0 - \rho, \rho_0 - \bar{\delta}), \\ \eta(\rho_0 - \rho) = 0, & \eta'(\rho_0 - \rho) = 0. \end{cases} \quad (3.12)$$

For η sufficiently close to $\eta = 0$ and $r \in (\rho_0 - \rho, \rho_0 - \bar{\delta})$, the quantity $\eta^p(1 - \frac{\mu}{(\rho_0 - r)^2} \eta^{1-p})$ is positive, hence $\eta(r)$ is increasing to the right in a small interval $(\rho_0 - \rho, \rho_0 - \tilde{\rho})$, for some $\tilde{\rho} \in (\bar{\delta}, \rho)$. As in (3.7) we obtain that $\eta(\rho_0 - \delta(x))$ is a local upper solution of (1.1) in $\Omega_{\rho} \setminus \Omega_{\tilde{\rho}}$.

If $\bar{U}(x) \leq \eta(\rho_0 - \tilde{\rho})$ for $\delta(x) = \tilde{\rho}$, there exists an eventually larger $\tilde{\rho}$, such that $\bar{U}(x) = \eta(\rho_0 - \tilde{\rho})$ for $\delta(x) = \tilde{\rho}$.

The following function is a (weak) global upper solution with dead core

$$\tilde{U}(x) := \begin{cases} \bar{U}(x), & x \in \Omega_{\tilde{\rho}}, \\ \eta(\rho_0 - \delta(x)), & x \in \Omega_{\rho} \setminus \Omega_{\tilde{\rho}}, \\ 0, & x \in \Omega \setminus \Omega_{\rho}. \end{cases}$$

If $\bar{U}(x) > \eta(\rho_0 - \tilde{\rho})$, we remark that for any $m \in (0, 1)$, $m\bar{U}(x)$ is an upper solution, hence we can choose m such that $m\bar{U}(x) = \eta(\rho_0 - \tilde{\rho})$ for $\delta(x) = \tilde{\rho}$ and conclude as above.

Concerning the lower solution we proceed as in (i), **(A)**. The conclusion follows as in case (i).

(B) $\mu < 0$.

We construct the upper solution as in (i), **(B)** solving (3.12) for $M = 0$. A positive solution exists by Lemma 2.1. No other changes are needed in the proof.

(iii) For any $\rho \in (0, \rho_0)$ we define the following subsets of Ω , say $(\Gamma_1)_{\rho} := \{x \in \Omega : d(x, \Gamma_1) < \rho\}$ and $(\Gamma_2)_{\rho} := \{x \in \Omega : d(x, \Gamma_2) < \rho\}$. ρ_0 is such that $(\Gamma_1)_{\rho_0}$ and $(\Gamma_2)_{\rho_0}$ are disjoint sets and $\Omega_{\rho_0} = (\Gamma_1)_{\rho_0} \cup (\Gamma_2)_{\rho_0}$.

In (ii) we constructed a solution u which vanishes in $\Omega \setminus \Omega_{\rho_0}$, hence the function

$$\tilde{u}(x) := \begin{cases} u(x), & x \in (\Gamma_1)_{\rho_0}, \\ 0, & x \in \Omega \setminus (\Gamma_1)_{\rho_0}, \end{cases}$$

is a solution and it has the behavior required in (iii). \square

Remark 3.1 1. In the proof of (i) we have constructed upper solutions and smaller nontrivial lower solutions which vanish in an interior set. The solutions we constructed lie between the upper and lower solutions, hence they might be strictly positive or have a dead core in some subsets of Ω .

2. Under the hypotheses in (iii), the number of solutions with a different qualitative behavior depends on the possible choices of the sets Γ_1 and Γ_2 , hence on the number of connected components of $\partial\Omega$. In particular the role of Γ_1 and Γ_2 can be exchanged.
3. We only required $\mu < \frac{1}{4}$ and not $\mu < C_H(\Omega)$. This weaker requirement is due to the fact that the Hardy constant of a thin set is $\frac{1}{4}$ and we mainly work in a thin neighborhood of the boundary.

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